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The contextuality of the possessed values

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Abstract. The mathematical structure of quantum mechanics is determined by two algorithms: quantization algorithm (QUAN) which says that an observed value of an observable Q is one of the eigenvalues q of the corresponding physical operator $\hat{Q}$, and the statistical algorithm (STAT) which says that the probability for the observed physical quantity $v_m(Q)$ to be q is $|\langle q|\psi\rangle|^2$. From these two algorithms we can derive the principle of statistical function composition (SFC), $\mathrm{Prob}(v_m(f(Q)) = r|\psi) = \mathrm{Prob}(v_m(Q) = f^{-1}(r)|\psi)$. With the assumption of the principle of function composition (FC) that the algebraic relations between observables of a given physical system are identical with the algebraic relations between the values possessed by the system before the observation is made, it could be explained why the result of measurement obeys SFC. According to Bell–KS theorem, FC is not compatible with quantum theory. We analyse, in this paper, the incompatibility of FC with quantum theory in detail, focusing on the analysis of Redhead (1989 *Incompleteness, Nonlocality and Realism* (Oxford: Oxford University Press)). Redhead assumes the faithful measurement principle (FM) in order to construct FC. However, we show, that FC can be derived without the help of FM.

1. Introduction

Many physical phenomena force physicists to accept the following two facts related to quantum physics:

- *QF1:* observed physical quantities are not allowed to be arbitrary but are restricted somehow.
- *QF2:* we cannot, in general, predict the result of measurement but can predict only the probability about the result of measurement.

When a physical system shows these two aspects, we describe this system through quantum theory which can be summarized by the following two algorithms, quantization algorithm (QUAN) and statistical algorithm (STAT) [1]:

- *QUAN:* a measured quantity of a physical observable Q is one of the eigenvalues of the corresponding operator $\hat{Q}$.
- *STAT:* when a physical system is in a state $|\psi\rangle$, the probability for the measured quantity to be q is $|\langle q|\psi\rangle|^2$.

However, we may require, the so-called minimal realism [2] in the measurement theory:

- *REALISM:* the measured quantity of a physical system is based on the properties of the system possessed before measurement is made.

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The programme to accomodate this REALISM into quantum theory is called the hidden-variables programme. However, Bell [3] and Kochen and Specker [4] showed that accomodation of REALISM into quantum theory is impossible, which is usually called the no-hidden-variables theorem. We now analyse the derivation of the Bell–KS theorem in detail and find out the problem in the hidden-variables programme. For this purpose, we follow the method introduced by Peres [5], Mermin [6–8] and Redhead [1].

2. The principle of statistical function composition

The wavefunction in a Schrödinger equation does not faithfully describe the physical properties of an individual particle. The conventional interpretation of quantum theory, therefore, does not allow us to assign precise physical quantities before measurement. On the other hand, according to the hidden-variables interpretation, we assume that there are hidden variables which determine all the physical quantities of individual particles. The hidden-variables programme, therefore, embodies quantum theory using the hidden variable λ. Are there any contradictions between quantum theory and the assumption of the existence of a hidden variable?

It is important whether or not there is a hidden variable. If there is a possibility of the existence of the hidden variable, the existing quantum theory is incomplete in some sense as discussed by Einstein *et al* [9]. However, if the assumption that there exists a hidden variable is not compatible with quantum theory, the hidden-variables programme cannot be realized. We analyse one of the no-hidden-variables theorems, Bell–KS theorem, and then find out what this theorem requires on the hidden-variables theory. For this purpose, we first investigate the mathematical theorem induced by the algorithms; quantization algorithm and statistical algorithm.

Let $\hat{Q}$ be a Hermitian operator defined on an N-dimensional Hilbert space. Consider its eigenstate $|q_i\rangle$ and eigenvalue q_i where i runs from 1 to N. When all of the q_i are different, we say $\hat{Q}$ is a nondegenerate or maximal operator. Two eigenstates corresponding to two different eigenvalues are orthogonal. Therefore, we get $\langle q_i|q_j\rangle = \delta_{ij}$. Introducing projection operators $\hat{P}_i = |q_i\rangle\langle q_i|$, $\hat{Q}$ can be expressed as $\hat{Q} = \sum_i q_i \hat{P}_i$. However, using $\hat{P}_i^n = \hat{P}_i$, a polynomial of $\hat{Q}$ can be written as $f(\hat{Q}) = \sum_i f(q_i)\hat{P}_i$. We now generalize this to an arbitrary function of the operator and define a function $\chi_{q_i}(q) = \delta_{q_i q}$. For this function, we get

$$\hat{P}_i = \chi_{q_i}(\hat{Q}). \tag{1}$$

According to the statistical algorithm, the probability for $v_m(Q)$, the measured value of the observable Q to be q_i, when a particle is in a state $|\psi\rangle$, is

$$\begin{aligned}\mathrm{Prob}(v_m(Q) = q_i|\psi) &= |\langle q_i|\psi\rangle|^2 = \langle q_i|\psi\rangle\langle\psi|q_i\rangle \\ &= \mathrm{tr}(\{|q_i\rangle\langle q_i|\}\{|\psi\rangle\langle\psi|\}) = \mathrm{tr}(\hat{P}_i\rho) = \mathrm{tr}(\chi_{q_i}(\hat{Q})\rho)\end{aligned} \tag{2}$$

where ρ is the state operator equal to $|\psi\rangle\langle\psi|$. From the relation $\chi_y(f(x)) = \chi_{f^{-1}(y)}(x)$ and using the equation (2), we get

$$\begin{aligned}\mathrm{Prob}(v_m(f(Q)) = r|\psi) &= \mathrm{tr}(\chi_r(f(\hat{Q}))\rho) \\ &= \mathrm{tr}(\chi_{f^{-1}(r)}(\hat{Q})\rho) = \mathrm{Prob}(v_m(Q) = f^{-1}(r)|\psi)\end{aligned} \tag{3}$$

where the function f is taken to have the inverse, or $f^{-1}(r) = \{x|f(x) = r\}$. As a result, we obtain the statistical functional composition principle (SFC),

$$\mathrm{Prob}(v_m(f(Q)) = r|\psi) = \mathrm{Prob}(v_m(Q) = f^{-1}(r)|\psi). \tag{4}$$

This SFC is the quantum mechanical restriction which should be satisfied by the distribution of measured values of the physical quantities. Now, how do we explain the fact that quantum phenomena requires SFC?

3. Hidden-variables theory and the functional composition principle

Let us investigate the way to explain this SFC in the hidden-variables theory. According to quantum theory, for any Hermitian operator $\hat{Q}$ there is a corresponding observable Q. First, hidden-variables theory should assume the following:

- *HV1:* a particle has $v_p(Q)$ as the possessed value of a physical observable Q before the measurement of Q is fulfilled.

Therefore, $v_p(Q)$ or $v_p(Q; |\psi\rangle, \lambda)$ is a real number, determined by the state of an individual particle described by $|\psi\rangle$ and λ, where λ denotes a hidden variable. According to the hidden-variables interpretation, a measured value $v_m(Q)$ is somehow related to the possessed value $v_p(Q)$. Then, what can be said about the relation between the measured value $v_m(Q)$ and the possessed value $v_p(Q)$? For that purpose, first we need to define the statistics of $v_p(Q)$. The statistics of $v_m(Q)$ can be identified with the distribution of the results of measurements. Then the statistics of $v_p(Q)$ can be defined as the distribution of the possessed values of the members of an ensemble. Here the ensemble is represented by $|\psi^E\rangle$ which corresponds to the wavefunction $|\psi\rangle$ and there is a one-to-one correspondence between $|\psi^E\rangle$ and $|\psi\rangle$. The probability that the possessed value is q is the number of elements in ensemble with q as $v_p(Q)$ divided by the total number of elements in the ensemble $|\psi^E\rangle$. For example, if 25% of the elements in the ensemble have q as the possessed value of Q, then $\text{Prob}(v_p(Q) = q|\psi^E)$ is equal to $\frac{1}{4}$.

In any case, according to QF2, quantum theory can definitely predict the statistics of the results of measurements. The Schrödinger equation gives the calulation of the statistics, which is what STAT says. The statement that quantum theory can predict the statistics of $v_m(Q)$ means that we physicists can obtain the statistics of $v_m(Q)$ through calulation on paper without performing any experiment in the laboratory. In principle, the value of $\text{Prob}(v_m(Q) = q|\psi)$ obtained by experiment is equal to $|\langle q|\psi\rangle|^2$ obtained by calculation of the Schrödinger equation. Therefore, the statistics of $v_m(Q)$ will not change in spite of influence of the measurement, and the statistics of $v_m(Q)$ can be defined before the measurement. If so, the statistics of $v_m(Q)$ should be expressed in terms of possessed values, since all information of the system before measurement is embodied in possessed values. Thus, the hidden-variables theorist must assume that the statistics of a possessed value defined independently of measurement is identical to the statistics of a measured value. Thus this hidden-variables interpreter's point of view can be summarized as follows:

- *HV2:* the statistics for the observed results after measurement can be identified with the statistics of the possessed values in an ensemble, or $\text{Prob}(v_m(Q) = q|\psi) = \text{Prob}(v_p(Q) = q|\psi^E)$.

Note that the assumption of HV2 does not force us to accept the following principle: the measured value is equal to the possessed value, or $v_m(Q) = v_p(Q)$. This principle is named the principle of faithful measurement (FM). This requirement of FM is proper, in some sense, because by measurement we mean to observe the properties of a particle (or a system) which were there before the measurement. From the realist's point of view, a particle has already had some properties before the measurement. Even though no measurement is performed, such properties are still there in reality. The measurement just plays the role of visualizing the properties of a given particle with no disturbances. However, we know that this requirement of FM is too strict. One of the objections to FM is as follows. Since quantum theory cannot predict the physical properties of a particular particle definitely, it is necessary to perform measurement in order to know the properties. But in the theory there is no the guarantee that the process of measurement does not disturb the properties of the particle. Therefore we will try not to commit to FM.

A case that HV2 holds without FM may be as follows. Suppose the system is in eigenstate of z-component of the spin. In this case, 50% of the ensemble has $\frac{1}{2}$ and another 50% has $-\frac{1}{2}$ as the possessed value of s_x. But as results of experiment, it may happen that only 90% of elements with $v_p(s_x) = \frac{1}{2}$ are measured as $v_m(s_x) = \frac{1}{2}$, eventually the rest 10% of results being $v_m(s_x) = \frac{1}{2}$ results from the elements with $v_p(s_x) = -\frac{1}{2}$. This is the case that HV2 holds without FM.

HV2 implies that for any observable Q if $\mathrm{Prob}(v_m(Q) = q|\psi) = 0$, then $\mathrm{Prob}(v_p(Q) = q|\psi^E) = 0$. Therefore, if we assume that when the probability for the possessed value to be q is zero, q cannot be a possessed value of the observable Q, and we can obtain the following VALUE rule. If $\mathrm{Prob}(v_m(Q) = q|\psi) = 0$, then $v_p(Q)$ cannot be q. This VALUE rule, however, holds only for the discrete spectral cases. For the case of a continuous spectrum we have to consider instead the probability density of probability itself. From VALUE we can derive the SPECTRUM rule that a possessed value of a observable Q should be one of the eigenvalues of the operator $\hat{Q}$. Now we further assume the following functional composition principle (FC) in addition to HV2:

- *FC:* the algebraic relations between observables are imposed exactly in the same way on the relations between the possessed values of the corresponding observables. In particular, $v_p(f(Q)) = f(v_p(Q))$ holds for any Q.

Using FC and HV2, we can derive the following statistical functional composition principle (SFC):

$$\begin{aligned}\mathrm{Prob}(v_m(f(Q)) = r|\psi) &= \mathrm{Prob}(v_p(f(Q)) = r|\psi^E)\\ &= \mathrm{Prob}(f(v_p(Q)) = r|\psi^E)\\ &= \mathrm{Prob}(v_p(Q) = f^{-1}(r)|\psi^E) = \mathrm{Prob}(v_m(Q) = f^{-1}(r)|\psi).\end{aligned} \tag{5}$$

Therefore, the hidden-variables theory accepting FC automatically allows SFC. The hidden-variables interpretation, that a particle carries possessed values in reality, provides a good explanation for the quantum mechanical experimental results when equipped with the assumption that the algebraic relations between the possessed values determine the quantum mechanical structure. However, FC does not come automatically from SFC, and consequently we do not know whether FC is compatible with the quantum mechanics. We now consider this question and will see that FC is not compatible with quantum mechanics.

4. Bell–KS theorem

What would happen if we require FC as a constraint on the possessed values? The answer to this question is the Bell–KS theorem [3, 4]. The conclusion of this theorem is that FC is not compatible with quantum mechanics. In order to show this conclusion, we need the following SUM rule. Let A and B be observables corresponding to operators $\hat{A}$ and $\hat{B}$ respectively, defined on an N-dimensional Hilbert space. If $\hat{A}$ and $\hat{B}$ commute, for any state vector the possessed value of an observable $A + B$ is equal to the sum of the possessed value of A and that of B. That is,

$$[\hat{A}, \hat{B}] = 0 \rightarrow v_p(A + B) = v_p(A) + v_p(B). \tag{6}$$

Before proving SUM, we need to derive a lemma.

Lemma. *If two operators $\hat{A}$ and $\hat{B}$ commute, there exist functions f, g and a nondegenerate operator $\hat{C}$ satisfying $\hat{A} = f(\hat{C})$, $\hat{B} = g(\hat{C})$.*

Proof. If two operators $\hat{A}$ and $\hat{B}$ commute, their eigenstates can be simultaneously defined. We denote them by $|a, b\rangle$ satisfying $\hat{A}|a, b\rangle = a|a, b\rangle$ and $\hat{B}|a, b\rangle = b|a, b\rangle$, and we assume they are orthonormalized, for orthonormalization is always possible. They form a complete set for the Hilbert space and can serve as a basis. We can express the two operators $\hat{A}$ and $\hat{B}$ as follows. $\hat{A} = \sum_a a|a, b\rangle\langle a, b|$ and $\hat{B} = \sum_b b|a, b\rangle\langle a, b|$. It is clear that both of the operators $\hat{A}$ and $\hat{B}$ are diagonalized at the same time in this basis. In the matrix form, $\hat{A}$ is represented by a diagonal matrix with its N eigenvalues $a_1, a_2, \ldots, a_N$ as diagonal elements, and likewise $\hat{B}$ is represented by a diagonal matrix with its N eigenvalues $b_1, b_2, \ldots, b_N$ as diagonal elements. Let $\hat{C}$ be an operator represented by a diagonal matrix with N distinct diagonal elements $c_1, c_2, \ldots, c_N$. It is obvious that there exists a function f satisfying $f(c_i) = a_i$, where $i = 1, 2, \ldots, N$. An obvious choice for f is an Nth degree polynomial with suitable coefficients. Similarly we can also find a function g satisfying $g(c_i) = b_i$,where $i = 1, 2, \ldots, N$. These two functions, defined originally as mappings from real numbers to real numbers, can be considered as mappings from the set of $N \times N$ matrices to the set of $N \times N$ matrices. In this sense, we get $\hat{A} = f(\hat{C})$, $\hat{B} = g(\hat{C})$. This proves the lemma. □

We now proceed to prove SUM using this lemma. For a given pair of commuting operators $\hat{A}$ and $\hat{B}$, we define a function h as the sum of f and g both of which satisfy $\hat{A} = f(\hat{C})$, $\hat{B} = g(\hat{C})$. With the assumption of FC, we finally get SUM:

$$\begin{aligned} v_p(A + B) &= v_p(f(C) + g(C)) = v_p(h(C)) = h(v_p(C)) \\ &= f(v_p(C)) + g(v_p(C)) = v_p(f(C)) + v_p(g(C)) = v_p(A) + v_p(B) \end{aligned} \tag{7}$$

where $f(C)$, $g(C)$ and $h(C)$ are the observables corresponding to operators $f(\hat{C})$, $g(\hat{C})$ and $h(\hat{C})$. By taking exactly the same procedure as in the proof of SUM, we can prove the PRODUCT rule. This rule for the possessed values of two commuting operators $\hat{A}$ and $\hat{B}$ is given as follows:

$$[\hat{A}, \hat{B}] = 0 \rightarrow v_p(AB) = v_p(A)v_p(B). \tag{8}$$

In summary, the assumption of FC automatically induces SUM and PRODUCT.

Now we introduce the Peres [5] and Mermin version [6, 8] of the proof that FC is not compatible with quantum physics. They used SUM and PRODUCT to prove it. More precisely speaking, they used two rules to derive a result which contradicts quantum mechanics. The proof is as follows. We consider a system composed of two spin-half particles in singlet state. Each particle carries a spin denoted by s_{im}, where index i is for the identification of the particle and m is for the component of spin. For example, s_{1y} indicates the y component of the spin of particle one. The eigenvalues of s_{im} are $\pm\frac{1}{2}$ for all i and m. In the proof, we use $\sigma_{im} = 2s_{im}$ for the sake of simplicity. So, the eigenvalues of σ_{im} are ± 1. We can think of σ_{im} as Pauli matrices for each i. Because the system is in singlet state, the system is in the eigenstate of $\sigma_{1x} + \sigma_{2x}$ with eigenvalue zero. According to SPECTRUM in section 3, the possessed value $v_p(\sigma_{1x} + \sigma_{2x})$ is also zero. Using SUM, we get $v_p(\sigma_{2x}) = -v_p(\sigma_{1x})$ and similar relations for the yth component and the zth component: $v_p(\sigma_{2y}) = -v_p(\sigma_{1y})$, $v_p(\sigma_{2z}) = -v_p(\sigma_{1z})$.

Now we consider two operators, $\sigma_{1x}\sigma_{2y}$ and $\sigma_{1y}\sigma_{2x}$. The two operators commute, as can easily be checked. We also know that $[\sigma_{1x}, \sigma_{1y}] = 0$ and $[\sigma_{1y}, \sigma_{2x}] = 0$. Using PRODUCT we get the following:

$$v_p(\sigma_{1x}\sigma_{2y}\sigma_{1y}\sigma_{2x}) = v_p(\sigma_{1x}\sigma_{2y})v_p(\sigma_{1y}\sigma_{2x})$$
$$= v_p(\sigma_{1x})v_p(\sigma_{2y})v_p(\sigma_{1y})v_p(\sigma_{2x}) = v_p(\sigma_{1x})^2 v_p(\sigma_{1y})^2 = 1. \tag{9}$$

We also have $\sigma_{ix}\sigma_{iy} = -\sigma_{iy}\sigma_{ix} = \mathrm{i}\sigma_{iz}$ for each i. And we have $\sigma_{1x}\sigma_{2y}\sigma_{1y}\sigma_{2x} = \sigma_{1x}\sigma_{1y}\sigma_{2y}\sigma_{2x} = \sigma_{1z}\sigma_{2z}$ and this gives us another relation:

$$v_p(\sigma_{1x}\sigma_{2y}\sigma_{1y}\sigma_{2x}) = v_p(\sigma_{1z}\sigma_{2z})$$
$$= v_p(\sigma_{1z})v_p(\sigma_{2z}) = -1. \tag{10}$$

Obviously this contradicts equation (9). This contradiction arises because SUM and PRODUCT are not compatible with quantum mechanics, which, in turn, means that FC is not compatible with quantum mechanics. In the next section, we further analyse how FC is not compatible with quantum mechanics.

5. Analysis of the principle of function composition

In addition to FC, if we accommodate FM that $v_m(Q) = v_p(Q)$ for any observable Q, we can easily obtain the following:

$$v_m(f(Q)) = v_p(f(Q)) = f(v_p(Q)) = f(v_m(Q)). \tag{11}$$

Intuitively, this result is quite acceptable on the ground that Q and $f(Q)$ can be measured simultaneously. But for the same reason that FC is not compatible with quantum mechanics, equation (11) also could not be accepted quantum mechanically. If FC implicitly contains FM as was assumed by Redhead [1], Bell–KS theorem forces us to discard FM. Then the hidden-variables interpretation should be carried out without assuming FM. But after discarding FM, what remains in the hidden-variables interpretation? To see this, we now analyse FC in different manner to Redhead [1].

According to HV1, a particle has possessed values $v_p(Q)$ for an arbitrary observable Q. We can construct new number $f(v_p(Q))$ for any invertible function f. The statistics of these numbers is determined solely by the statistics of $v_p(Q)$. As $f(v_p(Q)) = r$ and $v_p(Q) = f^{-1}(r)$ are equivalent statements, we have

$$\mathrm{Prob}(f(v_p(Q)) = r|\psi^E) = \mathrm{Prob}(v_p(f(Q)) = r|\psi^E) \tag{12}$$

using HV2 and SFC. Here we see that the statistics for $f(v_p(Q))$ is identical to the statistics for $v_p(f(Q))$, where $v_p(f(Q))$ is the possessed value of the observable $f(Q)$ corresponding to $f(\hat{Q})$. In other words, the probabilistic distribution for $f(v_p(Q))$ constructed by $v_p(Q)$ follows the statistical algorithm of the new operator $f(\hat{Q})$. Thus we make the following assumption:

- *REALITY:* if there exist real numbers satisfying probability distribution defined by the statistical algorithm of some Hermitian operator, there also exists a physical observable related to these numbers.

If this REALITY is accepted, there exists an observable $\tilde{f}(Q')$ related with the numbers $f(v_p(Q))$. The statistical algorithm of the observable $\tilde{f}(Q')$ is identical with that of the observable $f(Q)$ corresponding to the operator $f(\hat{Q})$. Now let us introduce another assumption called the correspondence rule (CR) as follows:

- *CR:* there is a one-to-one correspondence between an Hermitian operator $\hat{Q}$ and an observable Q.

If this CR assumption is accepted, there exists only one observable corresponding to the operator $f(\hat{Q})$. Thus $\tilde{f}(Q') = f(Q)$. And since the possessed values of the same observables are same, the possessed values of $f(Q)$ are equal to the possessed values of $\tilde{f}(Q')$. That is, $v_p(f(Q)) = f(v_p(Q))$, since $v_p(\tilde{f}(Q')) = f(v_p(Q))$ by construction of $\tilde{f}(Q')$. Therefore, we obtain the following FC:

$$v_p(f(Q)) = f(v_p(Q)).$$

According to the preceding analysis, FC can be derived from the following assumptions:

$$\text{HV1} \wedge \text{HV2} \wedge \text{REALITY} \wedge \text{CR} \rightarrow \text{FC}.$$

On the other hand, the fact that FC leads to contradiction implies that one should be discarding at least one of HV1, HV2, REALITY and CR. According to Bohr's point of view, or the Copenhagen interpretation, as observables are defined only by measurement, it is meaningless to suppose that a particle possesses certain numbers related with the observable Q before the measurement is made, so HV1 can be discarded. However, we can also discard some assumptions other than HV1. In fact, there exists a successful interpretation consistent with HV1, known as the de Broglie–Bohm interpretation [10, 11]. It should be noted that it is impossible to keep REALISM without the assumption HV1.

6. Contextuality

If FC is a condition which cannot be accepted quantum mechanically, the following conclusion can be drawn. Although operators $\hat{Q}$ and $\hat{A}$ are related with each other through $\hat{Q} = f(\hat{A})$, $v_p(Q)$ is not uniquely determined by $v_p(A)$, i.e., $v_p(Q) \neq f(v_p(A))$. This can be easily understood as follows. If for two commuting operators $\hat{A}$ and $\hat{B}$, there exist two functions f, g and a nondegerate operator $\hat{C}$ satisfying $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$. If f is a one-to-one function, $\hat{A}$ is nondegenerate and vice versa. Thus for nondegerate operator $\hat{A}$, f is a one-to-one function and has the inverse, which leads to $\hat{C} = f^{-1}(\hat{A})$. Consequently, $\hat{B} = g(\hat{C}) = g \circ f^{-1}(\hat{A})$, implying that $\hat{B}$ is expressed in terms of nondegenerate operator $\hat{A}$.

This means that an arbitrary operator $\hat{Q}$ can be expressed in terms of a nondegenerate operator commuting with $\hat{Q}$. But we may have two nondegenerate operators, $\hat{A}$ and $\hat{A}'$ such that $\hat{Q} = f(\hat{A}) = g(\hat{A}')$. If $\hat{Q}$ is also nondegenerate then $\hat{A} = f^{-1}(\hat{Q}) = f^{-1} \circ g(\hat{A}')$ and thus $\hat{A}\hat{A}' = \hat{A}'\hat{A}$. In fact, FC does not cause any trouble when $\hat{Q}$ is nondegenerate [12]. However, if $\hat{Q}$ is degenerate, we can find noncommuting nondegenerate operators $\hat{A}$ and $\hat{A}'$ such that $\hat{Q} = f(\hat{A}) = g(\hat{A}')$. In this case $v_p(Q) = v_p(f(A)) = f(v_p(A))$ and $v_p(Q) = v_p(g(A')) = g(v_p(A'))$ are simultaneously satisfied by FC. Consequently, FC requires $f(v_p(A)) = g(v_p(A'))$. However, for a degenerate operator $\hat{Q}$, $\hat{A}$ and $\hat{A}'$ may not commute with each other. If so, since A and A' cannot be measured simultaneously, we cannot claim that $f(v_p(A))$ and $g(v_p(A'))$ represent the possessed values of same observable of given ensemble member particle. In other words, there is no guarantee that $v_p(Q)$ in terms of $v_p(A)$ and $v_p(Q)$ in terms of $v_p(A')$ have the same value, which means that the possessed value of the degenerate observable can be different values on occasion.

The fact that the possessed values of an observable are not uniquely assigned may imply that the possessed value itself cannot be defined. This forces one to discard HV1. If so, the measured values vary with context of measuring apparatus in which the particle experiences, rather than determined by the possessed values. This philosophy of observables may be called epistemological contextualism (EC), which is supported by the Copenhagen school. According to this interpretation, the observable Q is a relational property defined by the measuring process.

However, if we want to support REALISM by adhering to HV1, we should discard either REALITY or CR. First, we discuss the validity of REALITY, which is open to doubt because we cannot claim that there exisits an observable $f(A)$ corresponding to the defined number $f(v_p(A))$. Consequently, the possessed values $v_p(Q)$ of an observable Q can be different from the values $f(v_p(A))$ constructed by the given function f. Namely, we cannot say that there necessarily exists an observable which has $f(v_p(A))$ as possessed values and corresponds to operator $f(\hat{A})$. Rather, $v_p(Q)$ of Q may happen to be equal to a certain value $g(v_p(A'))$ for some function g and $v_p(A')$ of another observable A'. Therefore, a true possessed value of observable Q may be different from a fake value $f(v_p(A))$ [13]. But it is very hard to understand that although Q and A are observables and $\hat{Q} = f(\hat{A})$, the quantity $f(A)$ may not be an observable.

So, alternatively, let us discard CR. We now suppose that the relation $\hat{Q} = f(\hat{A}) = g(\hat{A}')$ holds, where $\hat{Q}$ is any degenerate operator, $\hat{A}$ and $\hat{A}'$ are some noncommuting nondegenerate operators. If we discard CR, for nondegenerate operators $\hat{A}$ and $\hat{A}'$, there can be two observables Q_A, $Q_{A'}$ corresponding to an operator $\hat{Q}$. Therefore, the physical observables corresponding to one mathematical operator can be many, and have different possessed values in different contexts. Consequently, $v_p(Q_A) \neq v_p(Q_{A'})$ in general [14]. Let us call this philosophy of observables the ontological contextualism (OC). Now the observables are described by the relation between the corresponding operators and appropriate nondegenerate operators. Consequently, the context for making an observable a specific physical quantity is the context of measurement of a nondegenerate observable. In the experimental arrangement for the measurement of A, the observable corresponding to $\hat{Q}$ is Q_A, which implies that the measurement in this context gives $v_p(Q_A) = f(v_p(A))$ as the possessed value. The same applies to A'. But the fact that many observables correspond to one operator means that the structure of observables is more complicated than the mathematical structure of quantum mechanics. In other words, OC introduces too many observables into quantum mechanics. However, we understand the importance of observables corresponding to nondegenerate operators in the light of OC.

7. Conclusion

According to QUAN and STAT, possible values of an observable are determined by the principle of SFC. Since SFC can be derived from the principle of FC, the hidden-variables interpretation based on the assumption of FC gives a proper explanation for how the statistics of experimental results satisfies SFC. However, Bell–KS theorem states that the hidden-variables interpretation based on FC is incompatible with QUAN and STAT. We, without assuming the FM principle, tried to analyse FC to clarify why FC is incompatible with quantum mechanics. FC requires the following: (i) the validity of defining the possessed values (HV1); (ii) the statistical identity of the possessed values and the measured values (HV2); (iii) the one-to-one correspondence between the operator and the observable (CR); and (iv) the reality of an observable related to numbers obeying the statistical algorithm of a Hermitian operator (REALITY). Consequently, Bell–KS theorem requires that, among HV1+HV2, CR and REALITY, there is at least one that cannot be accepted to quantum mechanics. Therefore, the conclusion of Bell–KS theorem is that at least one of the following statements should be adopted: (i) the possessed value cannot be defined; (ii) there exists fake quantities; and (iii) there exist more observables than Hermitian operators. The hidden-variables interpreters should choose either (ii) or (iii), in order to support the minimal realism.

By analysing FC this way, we now understand why FM should be discarded. For instance, according to OC, measurement of an observable can have sense when the measuring context for nondegenerate observable is specified. Thus we do not know which observable corresponds to a given mathematical operator before the observable corresponding to the nondegenerate operator is prescribed. In other words, the possessed value of the degenerate observable generates after the context of measurement of the nondegenerate observable is specified. Therefore, for a degenerate operator, the possessed values and the measured values of the corresponding observable can be different.

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